

Extremal Kähler metrics

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Abstract. This paper is a survey of some recent progress on the study of Calabi’s extremal Kähler metrics. We first discuss the Yau-Tian-Donaldson conjecture relating the existence of extremal metrics to an algebro-geometric stability notion and we give some example settings where this conjecture has been established. We then turn to the question of what one expects when no extremal metric exists.

1. Introduction

A basic problem in differential geometry is to find “best” or “canonical” metrics on smooth manifolds. The most famous example is the classical uniformization theorem, which says that every closed 2-dimensional manifold admits a metric with constant curvature, and moreover this metric is essentially unique in its conformal class. Calabi’s introduction of extremal Kähler metrics [10] is an attempt at finding a higher dimensional generalization of this result, in the setting of Kähler geometry.

There are of course other ways in which one could attempt to generalize the uniformization theorem to higher dimensional manifolds. One possibility is the Yamabe problem in the context of conformal geometry. This says [31] that on a closed manifold of arbitrary dimension, every conformal class admits a metric of constant scalar curvature. Moreover this metric is often, but not always, unique up to scaling. A different generalization to 3-dimensional manifolds is given by Thurston’s geometrization conjecture, established by Perelman [41]. In this case the goal is to find metrics of constant curvature on a 3-manifold, but this is too ambitious. Instead it turns out that every 3-manifold can be decomposed into pieces each of which admits one of 8 model geometries.

The search for extremal Kähler metrics can be thought of as a complex analogue of the Yamabe problem, where we try to find canonical representatives of a given Kähler class, rather than a conformal class. In both cases the effect of restricting the space of metrics that we allow results in the problems reducing to scalar equations involving the conformal factor, and the Kähler potential, respectively. We will see, however, that in contrast with the Yamabe problem extremal metrics do not always exist, and in these cases one can hope to find a canonical “decomposition” of the manifold into pieces somewhat reminiscent of the geometrization of 3-manifolds.

In order to define extremal metrics, let M be a compact complex manifold of dimension n , equipped with a Kähler class $\Omega \in H^2(M, \mathbf{R})$. Denote by \mathcal{K}_Ω the set of Kähler metrics in the class Ω .

Definition 1. *An extremal metric is a critical point of the Calabi functional*

$$\begin{aligned} \text{Cal} : \mathcal{K}_\Omega &\rightarrow \mathbf{R} \\ \omega &\mapsto \int_M (S(\omega) - \underline{S})^2 \omega^n, \end{aligned} \tag{1}$$

where $S(\omega)$ is the scalar curvature of ω , and \underline{S} is the average of $S(\omega)$ with respect to the volume form ω^n . Note that \underline{S} is independent of the choice of $\omega \in \mathcal{K}_\Omega$.

Calabi [10] has shown that ω is an extremal metric if and only if the gradient $\nabla S(\omega)$ is a holomorphic vector field. Since most complex manifolds do not admit any non-trivial holomorphic vector fields, most extremal metrics are constant scalar curvature Kähler (cscK) metrics. A particularly important special case is when the first Chern class $c_1(M)$ is proportional to the Kähler class Ω . If $c_1(M) = \lambda\Omega$, and $\omega \in \mathcal{K}_\Omega$ is a cscK metric, then it follows that

$$\text{Ric}(\omega) = \lambda\omega, \tag{2}$$

and so ω is a Kähler-Einstein metric.

It is known that any two extremal metrics in a fixed Kähler class are isometric (see Chen-Tian [16]), which makes extremal metrics good candidates for being canonical metrics on Kähler manifolds. On the other hand, not every Kähler class admits an extremal metric, the first examples going back to Levine [32] of manifolds which do not admit extremal metrics in any Kähler class. The basic problems are therefore to understand which Kähler classes admit extremal metrics, and what we can say when no extremal metric exists.

The most interesting case of the existence question is when $\Omega = c_1(L)$ is the first Chern class of a line bundle, and consequently M is a projective manifold. In this case the Yau-Tian-Donaldson conjecture predicts that the existence of an extremal metric is related to the stability of the pair (M, L) in the sense of geometric invariant theory. In Section 2 we will discuss two such notions of stability: K-stability, and a slight refinement of it which we call \widehat{K} -stability.

As a consequence of work of Tian [54], Donaldson [18, 20], Mabuchi [34], Stoppa [44], Stoppa-Székelyhidi [45], Paul [39], Berman [7], and others, there are now many satisfactory results that show that the existence of an extremal metric implies various notions of stability. The converse direction, however, is largely open. In Section 3 we will discuss two results in this direction. One is the recent breakthrough of Chen-Donaldson-Sun [12] on Kähler-Einstein metrics with positive curvature, and the other is work of the author on extremal metrics on blowups.

Finally in Section 4 we turn to what is to be expected when no extremal metric exists, i.e. when a pair (M, L) is unstable. It is still a natural problem to try minimizing the Calabi functional in a Kähler class, and we will discuss a conjecture due to Donaldson relating this to finding the optimal way to destabilize (M, L) . We will give an example where this can be interpreted as the canonical decomposition of the manifold alluded to above.

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2. The Yau-Tian-Donaldson conjecture

It is a conjecture going back to Yau (see e.g. [60]) that if M is a Fano manifold, i.e. the anticanonical line bundle K_M^{-1} is ample, then M admits a Kähler-Einstein metric if and only if M is stable in the sense of geometric invariant theory. Tian [54] introduced the notion of K-stability as a precise candidate of such a stability condition and showed that it is necessary for the existence of a Kähler-Einstein metric. Donaldson [17, 19] generalized the conjecture to pairs (M, L) where $L \rightarrow M$ is an ample line bundle, not necessarily equal to the anticanonical bundle. More precisely, Donaldson formulated a more algebraic version of K-stability, and conjectured that K-stability of the pair (M, L) is equivalent to the existence of a cscK metric in the class $c_1(L)$. We start by giving a definition of Donaldson's version of K-stability.

Definition 2. A test-configuration for (M, L) of exponent r is a \mathbf{C}^* -equivariant, flat, polarized family $(\mathcal{M}, \mathcal{L})$ over \mathbf{C} , with generic fiber isomorphic to (M, L^r) .

The central fiber (M_0, L_0) of a test-configuration has an induced \mathbf{C}^* -action, and we write A_k for the infinitesimal generator of this action on $H^0(M_0, L_0^k)$. In other words, the eigenvalues of A_k are the weights of the action. There are expansions

$$\begin{aligned} \dim H^0(M_0, L_0^k) &= a_0 k^n + a_1 k^{n-1} + O(k^{n-2}) \\ \mathrm{Tr}(A_k) &= b_0 k^{n+1} + b_1 k^n + O(k^{n-1}) \\ \mathrm{Tr}(A_k^2) &= c_0 k^{n+2} + O(k^{n+1}). \end{aligned} \tag{3}$$

Definition 3. Given a test-configuration χ of exponent r for (M, L) as above, its Futaki invariant is defined to be

$$\mathrm{Fut}(\chi) = \frac{a_1 b_0 - a_0 b_1}{a_0^2}. \tag{4}$$

The norm of χ is defined by

$$\|\chi\|^2 = r^{-n-2} \left(c_0 - \frac{b_0^2}{a_0} \right), \tag{5}$$

where the factor involving r is used to make the norm unchanged if we replace L by a power.

With these preliminaries, we can give a definition of K-stability.

Definition 4. *The pair (M, L) is K-stable if $\text{Fut}(\chi) > 0$ for all test-configurations χ with $\|\chi\| > 0$.*

The condition $\|\chi\| > 0$ is required to rule out certain “trivial” test-configurations. An alternative definition by Li-Xu [33] requires \mathcal{M} to be a normal variety distinct from the product $M \times \mathbf{C}$, but the condition using the norm $\|\chi\|$ will be more natural below, when discussing filtrations.

The central conjecture in the field is the following.

Conjecture 5 (Yau-Tian-Donaldson). *Suppose that M has no non-zero holomorphic vector fields. Then M admits a cscK metric in $c_1(L)$ if and only if (M, L) is K-stable.*

When M admits holomorphic vector fields which can be lifted to L , then it is never K-stable according to the previous definition, since in that case one can find test-configurations with total space $\mathcal{M} = M \times \mathbf{C}$, with a non-trivial \mathbf{C}^* -action whose Futaki invariant is non-positive. In this case a variant of K-stability, called K-polystability, is used, which rules out such “product test-configurations” and is conjecturally equivalent to the existence of a cscK metric, even when M admits holomorphic vector fields. A further variant of K-stability, called relative K-stability, was defined by the author [49], and it is conjecturally related to the existence of extremal metrics. In relative K-stability one only considers test-configurations which are orthogonal to a maximal torus of automorphisms of M in a suitable sense.

Example 6. Let $(M, L) = (\mathbf{P}^1, \mathcal{O}(1))$. The family of conics $xz = ty^2$ for $t \in \mathbf{C}$ gives a test-configuration χ for (M, L) of exponent 2, degenerating a smooth conic into the union of two lines (see Figure 1). A small computation gives $\text{Fut}(\chi) = 1/8$. It is not surprising that this is positive, since the Fubini-Study metric on \mathbf{P}^1 has constant scalar curvature.

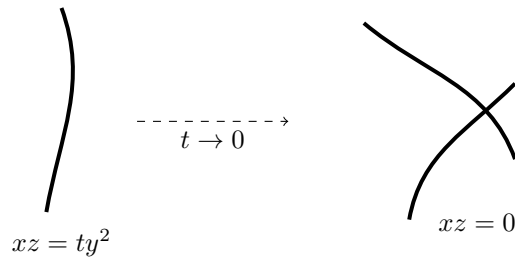


Figure 1. A test-configuration degenerating a conic into two lines.

Calculations in Apostolov-Calderbank-Gauduchon-Tønnesen-Friedman [1] suggest that K-stability might not be sufficient to ensure the existence of a cscK metric in general. Indeed they construct examples where the existence of an extremal metric is equivalent to the positivity of a certain function F on an interval (a, b) , while

relative K-stability only ensures that F is positive at rational points $(a, b) \cap \mathbf{Q}$. It is thus natural to try to work with a completion of the space of test-configurations in a suitable sense in order to detect when this function F vanishes at an irrational point. This motivates the the author's work [47] on filtrations.

Definition 7. Let $R = \bigoplus_{k \geq 0} H^0(M, L^k)$ denote the homogeneous coordinate ring of (M, L) . A filtration of R is a family of subspaces

$$\mathbf{C} = F_0 R \subset F_1 R \subset \dots \subset R, \quad (6)$$

satisfying

1. $(F_i R)(F_j R) \subset F_{i+j} R$,
2. If $s \in F_i R$, and s has degree k piece $s_k \in H^0(M, L^k)$, then $s_k \in F_i R$.
3. $R = \bigcup_{i \geq 0} F_i R$.

Witt Nyström [58] showed that every test-configuration for (M, L) gives rise to a filtration. In fact the Rees algebra of the filtration is the coordinate ring of the total space of the test-configuration. On the other hand given any filtration χ of the homogeneous coordinate ring of (M, L) , we obtain a flag of subspaces

$$\{0\} = F_0 R_r \subset F_1 R_r \subset \dots \subset R_r \quad (7)$$

of the degree r piece $R_r = H^0(M, L^r)$ for all $r > 0$. In turn, such a flag gives rise to a test-configuration of exponent r for (M, L) – by embedding M into projective space using a basis of $H^0(M, L^r)$, and acting by a \mathbf{C}^* -action whose weight filtration is given by our flag. Therefore any filtration χ induces a sequence of test-configurations $\chi^{(r)}$, where $\chi^{(r)}$ has exponent r . It is natural to think of χ as the limit of the $\chi^{(r)}$, and thus to define

$$\begin{aligned} \text{Fut}(\chi) &= \liminf_{r \rightarrow \infty} \text{Fut}(\chi^{(r)}) \\ \|\chi\| &= \lim_{r \rightarrow \infty} \|\chi^{(r)}\|, \end{aligned} \quad (8)$$

where the limit can be shown to exist. The main difference between filtrations arising from test-configurations, and general filtrations, is that the Rees algebras of the latter need not be finitely generated.

In terms of filtrations we define the following stability notion, which is stronger than K-stability.

Definition 8. The pair (M, L) is \widehat{K} -stable, if $\text{Fut}(\chi) > 0$ for all filtrations of the homogeneous coordinate ring of (M, L) satisfying $\|\chi\| > 0$.

In view of the examples of Apostolov-et. al. that we mentioned above, it may be that in the Yau-Tian-Donaldson conjecture one should assume \widehat{K} -stability instead of K-stability. One direction of this modified conjecture has been established by Boucksom and the author [47].

Theorem 9. *Suppose that M has no non-zero holomorphic vector fields. If M admits a cscK metric in $c_1(L)$, then (M, L) is \hat{K} -stable.*

The analogous result for K-stability was shown in Stoppa [44], building on work of Donaldson [20] which we will see in Theorem 14, and Arezzo-Pacard [2] which we will discuss in Section 3. In the proof of Theorem 9 the main additional ingredient is the use of the Okounkov body [37, 8, 29, 58]. Note that when $L = K_M^{-1}$, then related results were shown by Tian [54] and Paul-Tian [40]. It is likely that a result analogous to Theorem 9 can be shown for extremal metrics along the lines of [45].

3. Some existence results

In this section we discuss two special cases, where the Yau-Tian-Donaldson conjecture has been verified.

Kähler-Einstein metrics. We first focus on Kähler-Einstein metrics, i.e. when $c_1(M)$ is proportional to $c_1(L)$. When $c_1(M) = 0$, or $c_1(M) < 0$, then the celebrated work of Yau [59] implies that M admits a Kähler-Einstein metric, and a stability condition does not need to be assumed (see also Aubin [6] for the case when $c_1(M) < 0$).

In the remaining case, when $c_1(M) > 0$, i.e. M is Fano, it was known from early on (see e.g. Matsushima [35]) that a Kähler-Einstein metric does not always exist, and Yau conjectured that the existence is related to stability of M in the sense of geometric invariant theory. Tian [53] found all two-dimensional M which admit a Kähler-Einstein metric, and in [54] he formulated the notion of K-stability, which he conjectured to be equivalent to the existence of a Kähler-Einstein metric. The main difference between Tian's notion of K-stability and the one in Definition 4 is that Tian's version of K-stability only requires $\text{Fut}(\chi) > 0$ for very special types of test-configurations with only mild singularities. In particular their Futaki invariants can be computed differential geometrically using the formula Futaki [24] originally used to define his invariant. By the work of Li-Xu [33] it turns out that in the Fano case Tian's notion of K-stability is equivalent to the a priori stronger condition of Definition 4.

Recently, Chen-Donaldson-Sun [12, 13, 14, 15] have proved Conjecture 5 for Fano manifolds:

Theorem 10. *Suppose that M is a Fano manifold and (M, K_M^{-1}) is K-polystable. Then M admits a Kähler-Einstein metric.*

To construct a Kähler-Einstein metric, the continuity method is used, with a family of equations of the form

$$\text{Ric}(\omega_t) = t\omega_t + \frac{1-t}{m}[D], \quad (9)$$

where D is a smooth divisor in the linear system $|mK_M^{-1}|$, and $[D]$ denotes the current of integration. More precisely, a metric ω_t is a solution of (9) if $\text{Ric}(\omega_t) =$

$t\omega_t$ on $M \setminus D$, while ω_t has conical singularities along D with cone angle $\frac{2\pi}{m}(1-t)$. One then shows that Equation 9 can be solved for $t \in [t_0, T)$ for some $t_0, T > 0$ (see Donaldson [22] for the openness statement), and the question is what happens when $t \rightarrow T$.

One of the main results of the work of Chen-Donaldson-Sun is, roughly speaking, that along a subsequence the manifolds (M, ω_{t_i}) have a Gromov-Hausdorff limit W which is a \mathbf{Q} -Fano variety, such that the divisor $D \subset M$ converges to a Weil divisor Δ , and W admits a weak Kähler-Einstein metric with conical singularities along Δ (defined in an appropriate sense). Moreover, there are embeddings $\phi_i : M \rightarrow \mathbf{P}^N$ and $\phi : W \rightarrow \mathbf{P}^N$ into a sufficiently large projective space, such that the pairs $(\phi_i(M), \phi_i(D))$ converge to $(\phi(W), \phi(\Delta))$ in an algebro-geometric sense. In this case either $(\phi(W), \phi(\Delta))$ is in the $SL(N+1, \mathbf{C})$ -orbit of the $(\phi_i(M), \phi_i(D))$ in which case we can solve Equation (9) for $t = T$, or otherwise we can find a test-configuration for (M, D) with central fiber (W, Δ) to show that (M, K_M^{-1}) is not K-stable. Note that here one needs to extend the theory described in Section 2 to pairs (M, D) resulting in the notion of log K-stability [22].

The fact that a sequence of solutions to Equation (9) has a Gromov-Hausdorff limit which is a \mathbf{Q} -Fano variety originates in work of Tian [53] on the 2-dimensional case, and it is essentially equivalent to what Tian calls the “partial C^0 -estimate” being satisfied by such a sequence of solutions. This partial C^0 -estimate was first shown in dimensions greater than 2 by Donaldson-Sun [23] for sequences of Kähler-Einstein metrics, and their method has since been generalized to many other settings: Chen-Donaldson-Sun [14, 15] to solutions of (9); Phong-Song-Sturm [42] for sequences of Kähler-Ricci solitons; Tian-Zhang [55] along the Kähler-Ricci flow in dimensions at most 3; the author [48] along Aubin’s continuity method; Jiang [28] using only a lower bound for the Ricci curvature, in dimensions at most 3. Note that Tian’s original conjecture on the partial C^0 -estimate is still open in dimensions greater than 3 – namely we do not yet understand Gromov-Hausdorff limits of Fano manifolds under the assumption of only a positive lower bound on the Ricci curvature.

To close this subsection we mention a possible further result along the lines of Chen-Donaldson-Sun’s work. As we described above, if M does not admit a Kähler-Einstein metric, then a sequence of solutions to Equation (9) will converge to a weak conical Kähler-Einstein metric on a pair (W, Δ) as $t \rightarrow T$. Suppose $T < 1$. We can think of this metric as a suitable weak solution to the equation

$$\mathrm{Ric}(\omega_t) = t\omega_t + \frac{(1-t)}{m}[\Delta] \quad (10)$$

for $t = T$ on the space W . Since the pair (W, Δ) necessarily has a non-trivial automorphism group, we cannot expect to solve this equation for $t > T$, however it is reasonable to expect that we can still find weak conical Kähler-Ricci solitons, i.e. we can solve

$$\mathrm{Ric}(\omega_t) + L_{X_t}\omega_t = t\omega_t + \frac{(1-t)}{m}[\Delta], \quad (11)$$

for some range of values $t > T$, with suitable vector fields X_t fixing Δ . An extension of Chen-Donaldson-Sun’s work to Kähler-Ricci solitons, generalizing Phong-Song-

Sturm [42], could then be used to extract a limit (W_1, Δ_1) as $t \rightarrow T_1$, with yet another (weak) conical Kähler-Ricci soliton, and so on. Based on this heuristic argument we make the following conjecture.

Conjecture 11. *We can solve Equation (11) up to $t = 1$ by passing through finitely many singular times, changing the pair (W, Δ) each time. At $t = 1$ we obtain a \mathbf{Q} -Fano variety W_k admitting a weak Kähler-Ricci soliton. Moreover, there is a test-configuration for (M, K_M^{-1}) with central fiber W_k .*

A further natural expectation would be that the Kähler-Ricci soliton obtained in this way is related to the limiting behavior of the Kähler-Ricci flow on M . Indeed, according to the Hamilton-Tian conjecture (see Tian [54]), the Kähler-Ricci flow is expected to converge to a Kähler-Ricci soliton with mild singularities.

Blow-ups. Beyond the Kähler-Einstein case there are very few general existence results for cscK or extremal metrics. One example is the case of toric surfaces, where Conjecture 5 has been established by Donaldson [21], with an extension to extremal metrics by Chen-Li-Sheng [11]. In this section we will discuss a perturbative existence result for cscK metrics on blow-ups.

Suppose that ω is a cscK metric on a compact Kähler manifold M , and choose a point $p \in M$. For all sufficiently small $\epsilon > 0$ the class

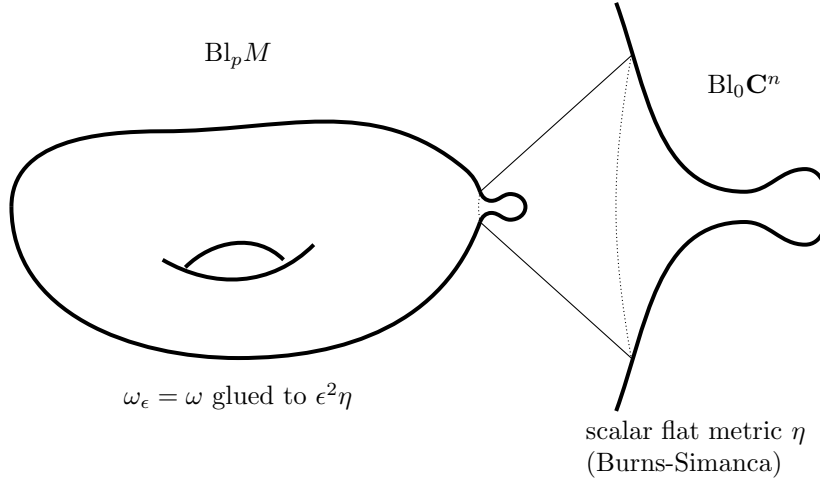
$$\Omega_\epsilon = \pi^*[\omega] - \epsilon^2[E] \quad (12)$$

is a Kähler class on the blowup $\text{Bl}_p M$, where $\pi : \text{Bl}_p M \rightarrow M$ is the blowdown map, and $[E]$ denotes the Poincaré dual of the exceptional divisor. A basic question, going back to work of LeBrun-Singer [30], is whether $\text{Bl}_p M$ admits a cscK (or extremal) metric in the class Ω_ϵ for sufficiently small ϵ . This problem was studied extensively by Arezzo-Pacard [2, 3], and Arezzo-Pacard-Singer [4]. See also Pacard [38] for a survey. The following is the most basic result in this direction.

Theorem 12 (Arezzo-Pacard [2]). *Suppose that M admits a cscK metric ω , and it admits no non-zero holomorphic vector fields. Then there is an $\epsilon_0 > 0$ such that $\text{Bl}_p M$ admits a cscK metric in the class Ω_ϵ for all $\epsilon \in (0, \epsilon_0)$.*

This result not only provides many new examples of cscK metrics, but it is also a key ingredient in the proofs of results such as Theorem 9. The construction of cscK metrics on blowups is a typical example of a gluing theorem in geometric analysis. First, one obtains a metric $\omega_\epsilon \in \Omega_\epsilon$ on the blowup, by gluing the metric ω to a scaled down version $\epsilon^2 \eta$ of the scalar flat Burns-Simanca [43] metric η on $\text{Bl}_0 \mathbf{C}^n$. This is shown in Figure 2. In a suitable weighted Hölder space the metric ω_ϵ is sufficiently close to having constant scalar curvature, that one can perturb it to a cscK metric using a contraction mapping argument, for sufficiently small ϵ .

When M admits non-zero holomorphic vector fields, then the problem becomes more subtle, since then $\text{Bl}_p M$ may not admit a cscK (or even extremal) metric for every point p . The problem was addressed by Arezzo-Pacard [3] and Arezzo-Pacard-Singer [4] in the case of extremal metrics, as well as the author [51, 46]. For the case of cscK metrics the sharpest result from [46] is as follows, showing that

Figure 2. The construction of the approximate metric ω_ϵ .

the Yau-Tian-Donaldson conjecture holds for the pair $(\text{Bl}_p M, \Omega_\epsilon)$ for sufficiently small ϵ .

Theorem 13. *Suppose that $\dim M > 2$, M admits a cscK metric ω , and $p \in M$. Then for sufficiently small $\epsilon > 0$, the blowup $\text{Bl}_p M$ admits a cscK metric in the class Ω_ϵ if and only if $(\text{Bl}_p M, \Omega_\epsilon)$ is K-polystable.*

For K-polystability to be defined algebraically, the class Ω_ϵ should be rational, but in fact a very weak version of K-polystability, which can be defined for Kähler manifolds, is sufficient in this theorem. Indeed what we can prove is that if $\text{Bl}_p M$ does not admit a cscK metric in the class Ω_ϵ for sufficiently small ϵ , then there is a \mathbf{C}^* -action λ on M such that if we let

$$q = \lim_{t \rightarrow 0} \lambda(t) \cdot p, \quad (13)$$

then the \mathbf{C}^* -action on $\text{Bl}_q M$ induced by λ has non-positive Futaki invariant. In other words when ϵ is sufficiently small, then it is enough to consider test-configurations for $\text{Bl}_p M$ which arise from one-parameter subgroups in the automorphism group of M . While there are also existence results for cscK metrics when $\dim M = 2$, and also for general extremal metrics, in these cases the precise relation with (relative) K-stability has not been established yet.

In the remainder of this section we will give a rough idea of the proof of these existence results. The basic ingredient is the existence of a scalar flat, asymptotically flat metric η on $\text{Bl}_0 \mathbf{C}^n$ due to Burns-Simanca [43], of the form

$$\eta = \sqrt{-1} \partial \bar{\partial} [|w|^2 + \psi(w)] \quad (14)$$

on $\mathbf{C}^n \setminus \{0\}$, where

$$\psi(w) = -|w|^{4-2n} + O(|w|^{2-2n}), \quad \text{as } |w| \rightarrow \infty \quad (15)$$

for $n > 2$. Under the change of variables $w = \epsilon^{-1}z$, we have

$$\epsilon^2 \eta = \sqrt{-1} \partial \bar{\partial} \left[|z|^2 + \epsilon^2 \psi(\epsilon^{-1}z) \right]. \quad (16)$$

At the same time there are local coordinates near $p \in M$ for which the metric ω is of the form

$$\omega = \sqrt{-1} \partial \bar{\partial} \left[|z|^2 + \phi(z) \right], \quad (17)$$

where $\phi(z) = O(|z|^4)$. One can then use cutoff functions to glue the metrics ω and $\epsilon^2 \eta$ on the level of Kähler potentials on the annular region $r_\epsilon < |z| < 2r_\epsilon$ for some small radius r_ϵ . The result is a metric $\omega_\epsilon \in \Omega_\epsilon$ on $\text{Bl}_p M$, which in a suitable weighted Hölder space is very close to having constant scalar curvature if ϵ is small. It is important here that η is scalar flat, since if it were not, then $\epsilon^2 \eta$ would have very large scalar curvature once ϵ is small.

When M has no holomorphic vector fields, then one can show that for sufficiently small ϵ this metric ω_ϵ can be perturbed to a cscK metric in its Kähler class, and this proves Theorem 12. Analytically the main ingredient in this proof is to show that the linearization of the scalar curvature operator is invertible, and to control the norm of its inverse in suitable Banach spaces as $\epsilon \rightarrow 0$.

The difficulty when M has holomorphic vector fields, or more precisely when when the Hamiltonian isometry group G of (M, ω) is non-trivial, is that the linearized operator will no longer be surjective, since its cokernel can be identified with the Lie algebra \mathfrak{g} of G . One way to overcome this issue is to try to solve a more general equation of the form

$$F(u, \xi) = 0, \quad (18)$$

where $\omega_\epsilon + \sqrt{-1} \partial \bar{\partial} u$ is a Kähler metric and $\xi \in \mathfrak{g}$. The operator F is constructed so that if $F(u, \xi) = 0$ and $\xi \in \mathfrak{g}_p$ is in the stabilizer of p , then $\omega_\epsilon + \sqrt{-1} \partial \bar{\partial} u$ is an extremal metric, which has constant scalar curvature if $\xi = 0$. At the same time the linearization of F is surjective. One can then show that for sufficiently small ϵ , for every point $p \in M$ we can find a solution $(u_{\epsilon,p}, \xi_{\epsilon,p})$ of the corresponding equation. The search for cscK metrics is then reduced to the finite dimensional problem of finding zeros of the map

$$\begin{aligned} \mu_\epsilon : M &\rightarrow \mathfrak{g} \\ p &\mapsto \xi_{\epsilon,p}, \end{aligned} \quad (19)$$

since if $\mu_\epsilon(p) = 0$, then we have found a cscK metric on $\text{Bl}_p M$ in the class Ω_ϵ . More generally to find extremal metrics we need to find p such that $\mu_\epsilon(p) \in \mathfrak{g}_p$.

At this point it becomes important to understand better what the map μ_ϵ is, and for this one needs to construct better approximate solutions than our crude attempt ω_ϵ above. In turn this requires more precise expansions of the metrics $\epsilon^2 \eta$ and ω than what we had in Equations (16) and (17). For the Burns-Simanca metric, according to Gauduchon [25] we have

$$\epsilon^2 \eta = \sqrt{-1} \partial \bar{\partial} \left[|z|^2 - d_0 \epsilon^{2m-2} |z|^{4-2m} + d_1 \epsilon^{2m} |z|^{2-2m} + O(\epsilon^{4m-4} |z|^{6-4m}) \right], \quad (20)$$

where $d_0, d_1 > 0$, while for the metric ω we have

$$\omega = \sqrt{-1}\partial\bar{\partial}\left[|z|^2 + A_4(z) + A_5(z) + O(|z|^6)\right], \quad (21)$$

where A_4 and A_5 are quartic and quintic expressions. Essentially A_4 is the curvature of ω at p , while A_5 is its covariant derivative at p . The way to obtain better approximate solutions than ω_ϵ is to preserve more terms in these expansions rather than multiplying them all with cutoff functions. In practice this involves modifying the metric ω on the punctured manifold $M \setminus \{p\}$ and $\epsilon^2\eta$ on $\text{Bl}_0\mathbf{C}^n$ to incorporate new terms in their Kähler potentials that are asymptotic to $-d_0\epsilon^{2m-2}|z|^{4-2m} + d_1\epsilon^{2m}|z|^{2-2m}$ and $A_4(z) + A_5(z)$ respectively.

The upshot is that we can obtain an expansion for μ_ϵ which is roughly of the form

$$\mu_\epsilon(p) = \mu(p) + \epsilon^2\Delta\mu(p) + O(\epsilon^\kappa) \quad (22)$$

for some $\kappa > 2$, where $\mu : M \rightarrow \mathfrak{g}$ is the moment map for the action of G on M , and $\Delta\mu$ is its Laplacian. At this point one can exploit the special structure of moment maps to show that if $\mu(p) + \epsilon^2\Delta\mu(p)$ is in the stabilizer \mathfrak{g}_p , and ϵ is sufficiently small, then there is a point $q \in G^c \cdot p$ in the orbit of p under the complexified group such that $\mu_\epsilon(q) \in \mathfrak{g}_q$. Since Bl_pM is biholomorphic to Bl_qM in this case, we end up with an extremal metric on Bl_pM . Under the K-polystability assumption this extremal metric is easily seen to have constant scalar curvature.

Finally, if $\mu(q) + \epsilon^2\Delta\mu(q) \notin \mathfrak{g}_q$ for any $q \in G^c \cdot p$ and sufficiently small ϵ , then the Kempf-Ness principle [36] relating moment maps to GIT stability can be exploited to find a \mathbf{C}^* -action on M which induces a destabilizing test-configuration for Bl_pM .

There are several interesting problems which we hope to address in future work.

1. One should extend Theorem 13 to the case when $\dim M = 2$ and to general extremal metric. In principle both of these extensions should follow from a more refined expansion of the function μ_ϵ than what we have in Equation (22), but it may be more practical to find a different, more direct approach.
2. Can one obtain similar existence results for blow-ups along higher dimensional submanifolds?
3. If M is an arbitrary compact Kähler manifold, is it possible to construct a cscK metric on the blowup of M in a sufficiently large number of points? This would be analogous to Taubes's result [52] on the existence of anti-self-dual metrics on the blowup of a 4-manifold in sufficiently many points. See Tipler [56] for a related result for toric surfaces, where iterated blowups are also allowed.

4. What if no extremal metric exists?

Even if M does not admit an extremal metric in a class $c_1(L)$, it is natural to try minimizing the Calabi functional. That this is closely related to the algebraic geometry of (M, L) is suggested by the following result, analogous to a theorem due to Atiyah-Bott [5] in the case of vector bundles.

Theorem 14 (Donaldson [20]). *Given a polarized manifold (M, L) , we have*

$$\inf_{\omega \in c_1(L)} \|S(\omega) - \underline{S}\|_{L^2} \geq \sup_{\chi} -c_n \frac{\text{Fut}(\chi)}{\|\chi\|}, \quad (23)$$

where the supremum runs over all test-configurations for (M, L) with $\|\chi\| > 0$, and c_n is an explicit dimensional constant.

Donaldson also conjectured that in fact equality holds in (23). When M admits an extremal metric $\omega_e \in c_1(L)$, then it is easy to check that

$$\|S(\omega_e) - \underline{S}\|_{L^2} = -c_n \frac{\text{Fut}(\chi_e)}{\|\chi_e\|}, \quad (24)$$

where χ_e is the product configuration built from the \mathbf{C}^* -action induced by $\nabla S(\omega_e)$. In other words equality holds in (23) in this case. When (M, L) admits no extremal metric, there is little known, except for the case of a ruled surface [50] where we were able to perform explicit constructions of metrics and test-configurations to realize equality in (23). Note that in the case of vector bundles the analogous conjecture is known to hold (i.e. equality in (23)) by Atiyah-Bott [5] over Riemann surfaces, and Jacob [27] in higher dimensions.

To describe our result, let Σ be a genus 2 curve, and $M = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(1))$, where $\mathcal{O}(1)$ denotes any degree one line bundle over Σ . For any real number $m > 0$, we have a Kähler class Ω_m on M , defined by

$$\Omega_m = [F] + m[S_0], \quad (25)$$

where $[F], [S_0]$ denote Poincaré duals to the homology classes of a fiber F and the zero section S_0 . Up to scaling we obtain all Kähler classes on M in this way. Depending on the value of m , in [50] we observed 3 qualitatively different behaviors of a minimizing sequence for the Calabi functional in Ω_m . There are explicitly computable numbers $0 < k_1 < k_2$ and minimizing sequences ω_i for the Calabi functional in Ω_m with the following properties:

1. When $m < k_1$, then the ω_i converge to the extremal metric in Ω_m whose existence was shown by Tønnesen-Friedman [57].
2. When $k_1 \leq m \leq k_2$ then suitable pointed limits of the ω_i are complete extremal metrics on $M \setminus S_0$ and $M \setminus S_\infty$. Here S_∞ is the infinity section, and the volumes of the two complete extremal metrics add up to the volumes of the ω_i .

3. When $m > k_2$, then suitable pointed limits of the ω_i are either $\Sigma \times \mathbf{R}$, or complete extremal metrics on $M \setminus S_0$ or $M \setminus S_\infty$. In the first case a circle fibration collapses, and the sum of the volumes of the two complete extremal metrics is strictly less than the volume of the ω_i . Figure 3 illustrates the behavior of the metrics ω_i when restricted to a \mathbf{P}^1 fiber.

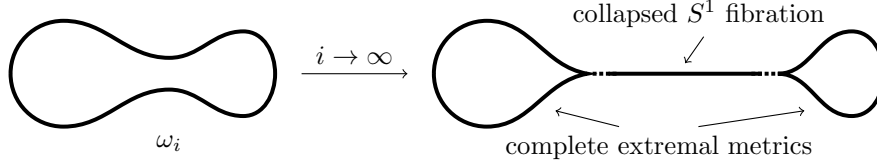


Figure 3. The fiber metrics of a minimizing sequence when $m > k_2$.

We interpret cases 2 and 3 as saying that a minimizing sequence breaks the manifold into several pieces. Some of the pieces admit complete extremal metrics, but others display more complicated collapsing behavior. Having such infinite diameter limits, and possible collapsing is in stark contrast with the case of Fano manifolds that we discussed in Section 3.

The sequences of metrics ω_i above can be written down explicitly using the momentum construction developed in detail by Hwang-Singer [26]. To show that these sequences actually minimize the Calabi energy, one needs to consider the right hand side of (23), and construct corresponding sequences of test-configurations χ_i such that

$$\lim_{i \rightarrow \infty} \|S(\omega_i) - \underline{S}\|_{L^2} = \lim_{i \rightarrow \infty} -c_n \frac{\text{Fut}(\chi_i)}{\|\chi_i\|}. \quad (26)$$

For this to make sense we need to assume that m is rational, so that a multiple of Ω_m is an integral class. Such a sequence χ_i can be constructed explicitly, and in the case when $m > k_2$, the exponents of the test-configurations χ_i tend to infinity with i . In other words, we need to embed M into larger and larger projective spaces in order to realize χ_i as a degeneration in projective space. The reason is that the central fiber of χ_i is a normal crossing divisor consisting of a chain of a large number of components isomorphic to M , with the infinity section of each meeting the zero section of the next one. The number of components goes to infinity with i . Figure 4 illustrates χ_i restricted to a \mathbf{P}^1 -fiber of M .

From Equation (26) together with Theorem 14 we obtain the following.

Theorem 15. *For the ruled surface M equality holds in Equation 23 for any polarization L .*

To conclude this section we point out that already in this example we cannot take a limit of the sequence χ_i in the space of test-configurations, because the exponents go to infinity. However there is a filtration χ , such that χ_i is the induced test-configuration of exponent i , and in this sense the limit of the χ_i exists as a filtration. This filtration achieves the supremum on the right hand side of (23)

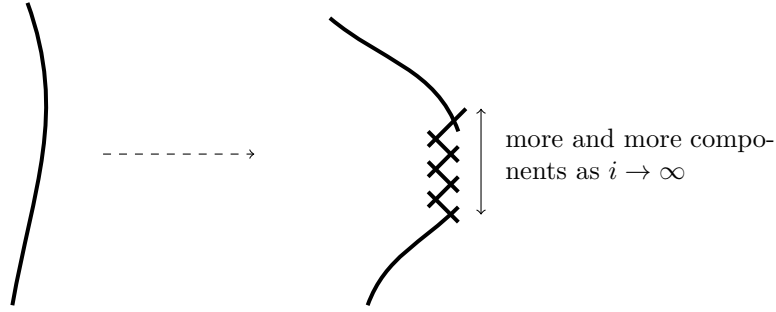


Figure 4. The test-configuration χ_i restricted to a \mathbf{P}^1 fiber.

and it is natural to ask whether such a “maximally destabilizing” filtration always exists. In view of the work of Bruasse-Teleman [9] this filtration, if it exists, should be viewed as analogous to the Harder-Narasimhan filtration of an unstable vector bundle.

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